Problem 16.12

Consider a semi-infinite string, fixed at the origin $x = 0$ and extending far out to the right. Let $f(\xi)$ be a function that is localized around the origin, such as the function of Figure 16.4(a). **(a)** Describe the wave given by the function $f(x + ct)$ for a large negative time t_o. **(b)** One way to solve for the subsequent motion of this wave on the semi-infinite string is called the **method of images** and is as follows: Consider the function $u = f(x + ct) - f(-x + ct)$. (The second term here is called the "image." Can you explain why?) Obviously this satisfies the wave equation for all x and t. Show that it coincides with the given wave of part (a) at the initial time t_o and everywhere on the semi-infinite string. Show also that it obeys the boundary condition that $u = 0$ at $x = 0$. **(c)** It is a fact that there is a unique wave that obeys the wave equation and any given initial and boundary conditions. Therefore the wave of part (b) is *the* solution for all times (on our semi-infinite string). Describe the motion on the semi-infinite string for all times.

Solution

Part (a)

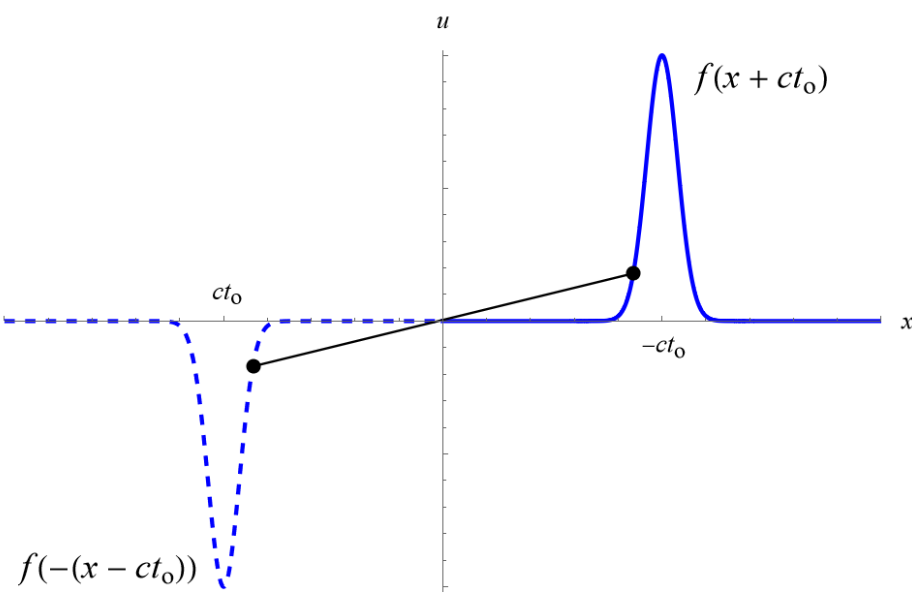
The wave given by $f(x + ct)$ at a large negative time t_o is $f(x + ct_o)$, which is the graph of $f(x)$ translated to the right by ct_o units. It's travelling to the left with speed c.

Part (b)

u is a superposition of two waves.

$$u(x,t) = \underbrace{f(x+ct)}_{\substack{\text{travelling to} \\ \text{the left}}} - \underbrace{f(-(x-ct))}_{\substack{\text{travelling to} \\ \text{the right}}}$$

The wave $f(-(x - ct))$ is said to be the image of $f(x + ct)$ because it's the reflection of $f(x + ct)$ on the other side of the origin.



Because the string only exists for $x > 0$, the image is an imaginary wave if the string were extended to $-\infty < x < \infty$. Its purpose is to make u zero at $x = 0$ at all times. Plugging in $t = t_0$ to u gives

$$u(x, t_0) = f(x + ct_0) - \underbrace{f(-x + ct_0)}_{=0} = f(x + ct_0).$$

The second term is zero because the wave is assumed to be localized around the origin, and $-x + ct_0$ is a large negative number while $x + ct_0$ is a small negative or small positive number. $f(-x + ct_0)$ is negligible compared to $f(x + ct_0)$. Plugging in $x = 0$ to u gives

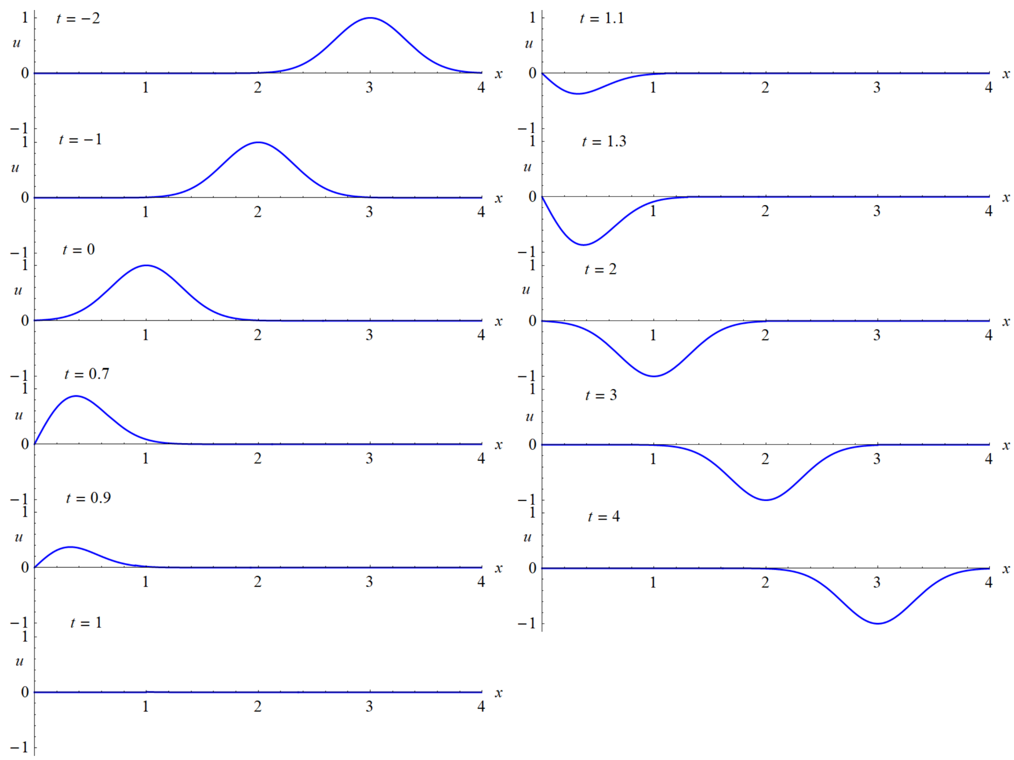
$$u(0, t_0) = f(ct_0) - f(ct_0) = 0,$$

so the boundary condition is satisfied.

Part (c)

As time increases from $t = t_0$, the two waves, $f(x + ct)$ and $f(-x + ct)$, approach each other. Once the tails of the curves meet, the waves begin to superimpose and interfere destructively with one another. When the waves are exactly on top of each other, they effectively cancel out. As times goes on further, the waves no longer influence each other, and what we see on $x > 0$ was the image that came from the negative x-axis. Below is an example solution for the case that

$$f(\xi) = e^{-5(\xi-1)^2}.$$



This solution $u = f(x + ct) - f(-x + ct)$ feels contrived, but it works because of the stipulation that f is localized around the origin. To see how to solve the wave equation on a semi-infinite interval from scratch using the method of reflection with no stipulations on f, see Strauss 3.2.3.